Stochastic Finite Element Methods with the Euclidean Degree for Partial Differential Equations with Random Inputs

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Abstract: In this paper, we construct a new implementation of stochastic finite element methods for partial differential equations with random inputs. The basis functions of generalized polynomial chaos are determined not by the usual notion of degree of a multivariate polynomial, but by the Euclidean degree. Then the corresponding linear combination of basis from the stochastic finite element methods is obtained, where the coefficient matrix is sparse and symmetric. In numerical experiments considering stochastic diffusion and Helmholtz equations, our approach with Euclidean degree of gPC basis achieves a better convergence rate than ones with total degree.

Key Words: stochastic finite element methods, partial differential equations with random inputs, Euclidean degree

1 INTRODUCTION

Partial differential equations (PDEs) are the basic frameworks for modeling an enormous variety of complex systems in both research and industry. Inputs of models, which are often formulated as random variables, describe the stochasticity of the system of interest and the output of the model becomes a random variable as well. Examples include the partially observable diffusion in filtering problems [1, 2], the fluid viscosity in incompressible flow problems [3, 4, 5, 6], the random surfaces in acoustic scattering [7]. The spectral methods have been investigated and developed to conduct the uncertainty quantification for PDEs where the spectral methods mainly include the stochastic collocations method and the stochastic Galerkin method. The main theoretical underpinning of the work presented herein is generally referred to as stochastic finite element methods (SFEMs) [8, 9, 10]. We combine the stochastic discretisation technologies with a discretisation of finite element method (FEM) and generalized polynomial chaos (gPC) expansion [2, 11, 12] in the stochastic space.

Based on the Wiener–Hermite polynomial chaos expansion [13], gPC methods seek to expand the random process (multi-variate) to a series in terms of a family of polynomial basis and then truncate the expansion to a finite version [12, 14]. Note that the retained terms are generally selected by making the usual notion of polynomial degree, namely the total degree, of every term less than a predetermined integer [3, 15]. However, there is a new concept about Euclidean degree of a multivariate polynomial, and it shows that the number of retained terms selected by Euclidean degree does not exceed that of based on total degree for the same accuracy [16, 17]. Our interest is to combine the gPC methods with Euclidean degree in stochastic space and propose a new SFEM, and the resulting method is highly efficient. On the application of two SPDEs problems, we demonstrate a good convergence rate of the retained terms selected by the Euclidean degree over the total degree.

This paper is organized as follows. In the next section, we introduce the SFEMs that contains our problem setting and the discretisation, and most importantly, the selection of basis functions with Euclidean degree in gPC. Section 3 provides empirical assessments of performance in two examples and the conclusions are in section 4.

2 STOCHASTIC FINITE ELEMENT METHODS

In this section, we review the stochastic finite element methods [14, 15, 18] for PDEs. The SFEMs include three major steps [19, 20, 21]: firstly, convert the partial differential problem into its variational formula;

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secondly, apply stochastic discretisation in both physical and stochastic spaces, which obtains a finite linear combination of stochastic basis functions; finally, adopt iterative method for fast solution schemes. This section mainly presents the problem setting and its discretisation, include the variational formulation of problem, discretisation and the selection of basis functions in the stochastic space.

2.1 Problem Setup

Let $D \subset \mathbb{R}^d$ denote a physical domain which is bounded, connected and with a polygonal boundary $\partial\mathbf{D}$, and $\boldsymbol{x} \in \mathbb{R}^d$ denote physical variables, where d is the dimension of physical space. Let $\boldsymbol{\xi} = [\xi_1, \ldots, \xi_M]$ be a random vector, and assume the random variables $\xi_1 \in \Gamma_1, \ldots, \xi_M \in \Gamma_M$ are mutually independent, where Γ_i is the image of the components ξ_i and M is the dimension of random vector. Then the image of $\boldsymbol{\xi}$ is given by $\Gamma = \Gamma_1 \times \ldots \times \Gamma_M$ and the joint probability density function of $\boldsymbol{\xi}$ is $\rho(\boldsymbol{\xi}) = \rho(\xi_1) \cdot \rho(\xi_2) \cdots \rho(\xi_M)$, where $\rho(\xi_i)$ is the probability density function of ξ_i for $i = 1, \ldots, M$. In this work, we consider the following PDE

$$\begin{aligned} -\nabla\cdot(a(\boldsymbol{x},\boldsymbol{\xi})\nabla u(\boldsymbol{x},\boldsymbol{\xi})) - \kappa^2(\boldsymbol{x},\boldsymbol{\xi})u(\boldsymbol{x},\boldsymbol{\xi}) &= f(\boldsymbol{x}), \\ \forall(\boldsymbol{x},\boldsymbol{\xi}) &\in \mathbf{D}\times\Gamma \\ u(\boldsymbol{x},\boldsymbol{\xi}) &= g(\boldsymbol{x}), \forall(\boldsymbol{x},\boldsymbol{\xi}) \in \partial\mathbf{D}_D \times \Gamma \\ a(\boldsymbol{x},\boldsymbol{\xi})\frac{\partial u}{\partial \mathbf{n}} &= h(\boldsymbol{x}), \forall(\boldsymbol{x},\boldsymbol{\xi}) \in \partial\mathbf{D}_N \times \Gamma \end{aligned} \quad (1)$$

where f is the source function, g, h specify the boundary conditions, they are all suitable deterministic functions, $\frac{\partial u}{\partial \mathbf{n}}$ denotes the outward normal derivative of u on the boundary, κ and a are coefficients as follows:

$$a(\boldsymbol{x},\xi) = \sum_{m=0}^{M} a_m(\boldsymbol{x})\xi_m, \quad \kappa(\boldsymbol{x},\xi) = \sum_{n=0}^{M} k_n(\boldsymbol{x})\xi_n$$

where $\{a_m(\boldsymbol{x})\}_{m=0}^{M}$ and $\{k_n(\boldsymbol{x})\}_{n=0}^{M}$ are real functions in physical domain, $a_0(\boldsymbol{x}) = 1$ and $\xi_0 = 1$. Moreover, we assume that the Dirichlet boundary $\partial\mathbf{D}$ and the Neumann boundary $\mathbf{D}_N$ satisfying

$$\partial\mathbf{D}_D \cup \partial\mathbf{D}_N = \partial\mathbf{D}, \quad \partial\mathbf{D}_D \cap \partial\mathbf{D}_N = \emptyset.$$

2.1.1 Variational Formulation

For simplicity, we assume the derivation of SFEMs for equation (1) satisfying homogeneous Dirichlet boundary conditions only. For the non-homogeneous cases, there are more detailed derivations in [9, 11, 22].
Let us denote $L^2(D)$ and $L^2_\rho(\Gamma)$ as the Hilbert spaces

$$L^2(D) = \Big\{ v(\boldsymbol{x}) : D \to \mathbb{R} \;\Big|\; \int_D v^2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} < \infty \Big\},$$
$$L^2_\rho(\Gamma) = \Big\{ g(\boldsymbol{\xi}) : \Gamma \to \mathbb{R} \;\Big|\; \int_\Gamma \rho(\boldsymbol{\xi}) g^2(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} < \infty \Big\}.$$

The inner products of these Hilbert spaces are given by

$$\langle v_1(\boldsymbol{x}), v_2(\boldsymbol{x})\rangle_{L^2(D)} = \int_D v_1(\boldsymbol{x}) v_2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$
$$\langle g_1(\boldsymbol{\xi}), g_2(\boldsymbol{\xi})\rangle_{L^2_\rho(\Gamma)} = \int_\Gamma \rho(\boldsymbol{\xi}) g_1(\boldsymbol{\xi}) g_2(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi},$$

and the according norm are

$$\|v\|_{L^2(D)} = \left(\int_D v^2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \right)^{1/2},$$
$$\|g\|_{L^2_\rho(\Gamma)} = \left(\int_\Gamma \rho(\boldsymbol{\xi}) g^2(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \right)^{1/2}.$$

Next, the tensor space of $L^2(D)$ and $L^2_\rho(\Gamma)$ is defined as

$$L^2(D) \otimes L^2_\rho(\Gamma) = \Big\{ w(\boldsymbol{x},\boldsymbol{\xi}) \;\Big|\; w(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{i=1}^{n} v_i(\boldsymbol{x}) g_i(\boldsymbol{\xi}), \; v_i(\boldsymbol{x}) \in L^2(D), g_i(\boldsymbol{\xi}) \in L^2_\rho(\Gamma), n \in \mathbb{N}^+$$

which is equipped with the inner product

$$\langle w_1(\boldsymbol{x},\boldsymbol{\xi}), w_2(\boldsymbol{x},\boldsymbol{\xi})\rangle_\otimes = \sum_{i,j} \langle v_i(\boldsymbol{x}), v_j(\boldsymbol{x})\rangle_{L^2(D)} \langle g_i(\boldsymbol{\xi}), g_j(\boldsymbol{\xi})$$

Apparently, the inner product of $L^2(D) \otimes L^2_\rho(\Gamma)$ can be represented as

$$\langle w_1(\boldsymbol{x},\boldsymbol{\xi}), w_2(\boldsymbol{x},\boldsymbol{\xi})\rangle_\otimes = \int_\Gamma \int_D \rho(\boldsymbol{\xi}) w_1(\boldsymbol{x},\boldsymbol{\xi}) w_2(\boldsymbol{x},\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}$$

Furthermore, we define the constrained physical domain as

$$H^1_0(D) := \Big\{ v \in H^1(D) \;\Big|\; v|_{\partial D} = 0 \Big\},$$

where $H^1(D)$ is a Sobolev space defined as

$$H^1(D) = \Big\{ v \in L^2(D) \;\Big|\; \frac{\partial v}{\partial x_i} \in L^2(D), \;\; i \in \{1, \ldots, d\} \Big\}.$$

Then the solution space can be written as

$$\begin{aligned} W &= H^1_0(D) \otimes L^2_\rho(\Gamma) \\ &= \Big\{ w(\boldsymbol{x},\boldsymbol{\xi}) \;\Big|\; w(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{i=1}^{n} v_i(\boldsymbol{x}) g_i(\boldsymbol{\xi}), \\ &\quad v_i(\boldsymbol{x}) \in H^1_0(D), g_i(\boldsymbol{\xi}) \in L^2_\rho(\Gamma), n \in \mathbb{N}^+ \Big\}. \end{aligned}$$

Finally, the variational formula of (1) can be restated as: find $u(\boldsymbol{x},\boldsymbol{\xi}) \in W$ such that

$$\langle a\nabla u, \nabla w\rangle_\otimes - \langle \kappa u, \kappa w\rangle_\otimes = \langle f, w\rangle_\otimes, \forall w(\boldsymbol{x},\boldsymbol{\xi}) \in W. \quad (2)$$

2.1.2 Discretisation

In order to discretize (2), we now seek for a finite-dimensional subspace to approximate W. The discretisation formulation of the physical space $H^1_0(D)$ and the random space $L^2_\rho(\Gamma)$ can be written as

$$V_h = \mathrm{span}\{v_s(\boldsymbol{x})\}_{s=1}^{N_x} \subset H^1_0(D),$$
$$S = \mathrm{span}\{\Phi_j(\boldsymbol{\xi})\}_{j=1}^{N_\xi} \subset L^2_\rho(\Gamma),$$

where $v_s(\boldsymbol{x}), \Phi_j(\boldsymbol{\xi})$ are the basis functions of $H_0^1(D)$ and $L_\rho^2(\Gamma)$ respectively. Therefore, a finite-dimensional subspace W_h can be defined as

$$W_h = V_h \otimes S = \operatorname{span}\Big\{ v(\boldsymbol{x})\Phi(\boldsymbol{\xi}) \;\Big|\; v \in V_h, \Phi \in S \Big\}.$$

The SFEM seeks an approximation $u^{ap}(\boldsymbol{x}, \boldsymbol{\xi}) \in W_h$ satisfying

$$\langle a\nabla u^{ap}, \nabla w\rangle_\otimes - \langle \kappa u^{ap}, \kappa w\rangle_\otimes = \langle f, w\rangle_\otimes, \tag{3}$$

where the finite-dimensional solution $u^{ap}(\boldsymbol{x}, \boldsymbol{\xi})$ can be represented as

$$u^{ap}(\boldsymbol{x}, \boldsymbol{\xi}) := \sum_{j=1}^{N_\xi}\sum_{s=1}^{N_x} u_{js} v_s(\boldsymbol{x})\Phi_j(\boldsymbol{\xi}), \tag{4}$$

where u_{js} is the target unknown coefficient.
Combining (4) with(3), the discrete variational formulation of (2) can be written as

$$\boldsymbol{A}\overline{\boldsymbol{u}} = \boldsymbol{b}, \tag{5}$$

where

$$\boldsymbol{A} = \sum_{l=0}^{M} \boldsymbol{G}_{l0} \otimes \boldsymbol{L}_l - \sum_{l=0}^{M}\sum_{n=0}^{M} \boldsymbol{G}_{ln} \otimes \boldsymbol{Q}_{ln}, \quad \boldsymbol{b} = \boldsymbol{h} \otimes \boldsymbol{f}.$$

In equation (5), $\otimes$ is Kronecker product, $\boldsymbol{G}_{ln} \in \mathbb{R}^{N_\xi \times N_x}$ are stochastic matrixes, $\boldsymbol{h}$ denotes a N_ξ-column vector, and they are defined as

$$\begin{aligned}
\boldsymbol{G}_{ln}(j,i) &= \langle \xi_l\Phi_j, \xi_n\Phi_i\rangle_{L_\rho^2(\Gamma)} \\
\boldsymbol{L}_l(s,t) &= \langle a_l\nabla v_s, \nabla v_t\rangle_{L^2(D)}, \\
\boldsymbol{Q}_{ln}(s,t) &= \langle \kappa_l v_s, \kappa_n v_t\rangle_{L^2(D)}, \\
\boldsymbol{h}(i) &= \langle \Phi_i, 1\rangle_{L_\rho^2(\Gamma)}, \\
\boldsymbol{f}(t) &= \langle f, v_t\rangle_{L^2(D)}
\end{aligned} \tag{6}$$

where $l, n \in \{0, 1, \ldots, M\}$, $j, i \in \{1, \ldots, N_\xi\}$, $s, t \in \{1, \ldots, N_x\}$, and $\overline{\boldsymbol{u}}$ is a $(N_x N_\xi)$-column vector of which the definition is

$$\overline{\boldsymbol{u}} = \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_{N_\xi} \end{bmatrix}, \text{where } \boldsymbol{u}_j = \begin{bmatrix} u_{1j} \\ \vdots \\ u_{N_x j} \end{bmatrix}, j \in \{1, \ldots, N_\xi\}.$$

In fact, the matrix $\boldsymbol{A}$ of linear system (5) can be presented as a form of block-structure

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{1,1} & \boldsymbol{A}_{1,2} & \cdots & \boldsymbol{A}_{1,N_\xi} \\ \boldsymbol{A}_{2,1} & \boldsymbol{A}_{2,2} & \cdots & \boldsymbol{A}_{2,N_\xi} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}_{N_\xi,1} & \boldsymbol{A}_{N_\xi,2} & \cdots & \boldsymbol{A}_{N_\xi,N_\xi} \end{pmatrix}, \tag{7}$$

where each block $\boldsymbol{A}_{j,k}$ is a $N_x \times N_x$ matrix and $j, k \in \{1, \ldots, N_\xi\}$.
We can obtain the numerical solution $u^{ap}(\boldsymbol{x}, \boldsymbol{\xi})$ by solving the system (5). After that, the mean and variance of exact solution $u(\boldsymbol{x}, \boldsymbol{\xi})$ can also be approximated as

$$\begin{aligned}
\mathbb{E}\big[u(\boldsymbol{x}, \boldsymbol{\xi})\big] &\approx \mathbb{E}\big[u^{ap}(\boldsymbol{x}, \boldsymbol{\xi})\big], \\
\mathbb{V}\big[u(\boldsymbol{x}, \boldsymbol{\xi})\big] &\approx \mathbb{V}\big[u^{ap}(\boldsymbol{x}, \boldsymbol{\xi})\big].
\end{aligned}$$

2.2 Selection of Basis Functions in Stochastic Space

As the derivation process shows, the way of selecting basis functions directly determine the performance of approximation. In this section, we discuss the selection of basis functions in the stochastic space and introduce the gPC methods with Euclidean degree [16, 17].

2.2.1 gPC with Euclidean degree

Let $\xi_1, \ldots, \xi_M$ be independent random variables with probability density functions $\rho_1, \ldots, \rho_M$, the image of them are $\Gamma_1, \ldots, \Gamma_M$ respectively. and the random vector $\boldsymbol{\xi}$ is defined as $\boldsymbol{\xi} = [\xi_1, \ldots, \xi_M]^{\mathrm{T}}$. Obviously, the probability density function of $\boldsymbol{\xi}$ is $\rho(\boldsymbol{\xi}) = \rho_1(\xi_1) \cdots \rho_M(\xi_M)$, and the image is denoted as $\Gamma = \Gamma_1 \times \cdots \times \Gamma_M$.
For an univariate ξ_i, the gPC basis functions are the orthogonal polynomials [14] satisfying

$$\mathbb{E}\big[\phi_j(\xi_i)\phi_k(\xi_i)\big] = \int_\Gamma \rho_i(\xi_i)\phi_j(\xi_i)\phi_k(\xi_i)\mathrm{d}\xi_i = \delta_{jk}, j, k \in \mathbb{N}^+$$

where j, k are the order of the polynomials ϕ_j and ϕ_k, δ_{jk} is Kronecker's delta and $\mathbb{N}^+$ is the set of non-negative integers. Since the random variables are independent, the gPC basis functions of $\boldsymbol{\xi}$ can be written as

$$\Phi_{\boldsymbol{k}}(\boldsymbol{\xi}) = \phi_{k_1}(\xi_1) \cdots \phi_{k_M}(\xi_M), \quad k_1, \ldots, k_M \in \mathbb{N}^+,$$

where $\boldsymbol{k} = [k_1, \ldots, k_M]^{\mathrm{T}}$ is a multi-index. Denote the single-index corresponding to $\boldsymbol{k}$ as k, and it shows that

$$\mathbb{E}\big[\Phi_j(\boldsymbol{\xi})\Phi_{\boldsymbol{k}}(\boldsymbol{\xi})\big] = \int_\Gamma \rho(\boldsymbol{\xi})\Phi_j(\boldsymbol{\xi})\Phi_{\boldsymbol{k}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \delta_{\boldsymbol{jk}},$$

where

$$\delta_{\boldsymbol{jk}} = \delta_{j_1 k_1} \cdots \delta_{j_s k_s} = \delta_{jk}.$$

For a multi-variable function $u(\boldsymbol{\xi})$, it can be approximated by a truncated gPC expansion

$$u(\boldsymbol{\xi}) \approx \sum_{\boldsymbol{k} \in \Xi} a_{\boldsymbol{k}} \Phi_{\boldsymbol{k}}(\boldsymbol{\xi}), \tag{8}$$

where Ξ is a multi-index set, $\Phi_{\boldsymbol{k}}$ is one of the gPC basis functions, and $a_{\boldsymbol{k}}$ is the expansion coefficient.
Since every multi-index is corresponding to a gPC basis function, the basis functions, retained in the truncated gPC expansion (8), is determined when the multi-index set Ξ is set up. For notational convenience, we construct the one-to-one mapping connecting the multi-index basis functions with the single-index ones. Table 1 shows an example of the graded lexicographic ordering with $M = 4$ (M is the dimension of random vector). Obviously, how to choose Ξ is an important issue. Next, we introduce different ways to select basis functions.

Table 1: Example of a graded lexicographic ordering with $M = 4$

$\lvert\boldsymbol{k}\rvert$	single-index k	multi-index $\boldsymbol{k}$
0	1	(0,0,0,0)
1	2	(0,0,0,1)
	3	(0,0,1,0)
	4	(0,1,0,0)
	5	(1,0,0,0)
2	6	(0,0,0,2)
	7	(0,0,1,1)
	...	...

For a monomial $\xi_1^{k_1} \dots \xi_M^{k_M}$, there are usually three kinds of degree, total degree, Euclidean degree and maximal degree [17], respectively as

$$\begin{aligned} \text{Total degree: } & d_T(\boldsymbol{k}) = \|\boldsymbol{k}\|_1, \\ \text{Euclidean degree: } & d_E(\boldsymbol{k}) = \|\boldsymbol{k}\|_2, \\ \text{Maximal degree: } & d_M(\boldsymbol{k}) = \|\boldsymbol{k}\|_\infty, \end{aligned}$$

where $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are the 1-, 2- and ∞-norms of the M-vector $\boldsymbol{k} = [k_1, \dots, k_M]^{\mathrm{T}}$ accordingly. For a general multivariate polynomial, its degree is defined as the maximum of the degrees in all nonzero monomial components. Furthermore, the definition of total degree and maximal degree can be found in many textbooks and papers [17, 23, 24], and the Euclidean degree in [16, 17].

By the definitions above, it is clear that the degree of multi-variable polynomial $\Phi_{\boldsymbol{k}}(\boldsymbol{\xi})$ is the same as that of $\xi_1^{k_1} \cdots \xi_M^{k_M}$. The multi-index set Ξ can be selected based on one kind of the degrees. For total degree, the selected multi-index set is denoted as Ξ_T^p, where Ξ_T^p is defined as

$$\Xi_T^p = \{\boldsymbol{k} | \boldsymbol{k} \in \mathbb{R}^M, k_1, \dots, k_M \in \mathbb{N}^+,\ d_T(\boldsymbol{k}) \le p\}.$$

Similarly, for Euclidean degree and maximal degree, the selected multi-index set are denoted as Ξ_E^p and Ξ_M^p separately, and they are defined as

$$\begin{aligned} \Xi_E^p &= \{\boldsymbol{k} | \boldsymbol{k} \in \mathbb{R}^M, k_1, \dots, k_M \in \mathbb{N}^+,\ d_E(\boldsymbol{k}) \le p\}. \\ \Xi_M^p &= \{\boldsymbol{k} | \boldsymbol{k} \in \mathbb{R}^M, k_1, \dots, k_M \in \mathbb{N}^+,\ d_M(\boldsymbol{k}) \le p\}. \end{aligned}$$

To further elaborate the construction of random space S, under the selection of basis functions with Euclidean degree, let us take a example of uniform random variables with $M = 2$ and $p = 3$. That means, $\boldsymbol{\xi} = [\xi_1, \xi_2]^{\mathrm{T}}$, and S is a set of two-dimensional Legendre polynomial chaos [14] whose degree is no more than 3. Every basis function in S is equipped with a multi-index $\boldsymbol{k} = (k_1, k_2)$, where k_1, k_2 are the degrees of polynomials in ξ_1 and ξ_2 respectively. Denote the three spaces as S^T, S^E, S^M associated with the selection of basis function with total, Euclidean, and maximal degree separately. Given the uni-variate Legendre polynomial of degrees 0,1,2,3 are $P_0(\xi_i) = 1, P_1(\xi_i) = \xi_i, P_2(\xi_i) = \frac{3}{2}{\xi_i}^2 - \frac{1}{2}, P_3(\xi_i) = \frac{5}{2}{\xi_i}^3 - \frac{3}{2}\xi_i$ for $i \in \{1, 2\}$, then the multi-index sets of S^T, S^E, S^M are

$$\begin{aligned} \Xi^T =& \{(0,0), (0,1), (1,0), (0,2), (1,1), (2,0), (0,3), (1,2), \\ & (2,1), (3,0)\}, \\ \Xi^E =& \{(0,0), (0,1), (1,0), (0,2), (1,1), (2,0), (0,3), (1,2), \\ & (2,1), (3,0), (2,2)\}, \\ \Xi^M =& \{(0,0), (0,1), (1,0), (0,2), (1,1), (2,0), (0,3), (1,2), \\ & (2,1), (3,0), (1,3), (2,2), (3,1), (2,3), (3,2), (3,3)\}. \end{aligned}$$

According to the multi-index sets, let us give the case of space S^E here

$$\begin{aligned} S^T =& \operatorname{span}\{\Phi_j(\boldsymbol{\xi})\}_{j=1}^{10} \\ =& \Big\{1, \xi_2, \xi_1, \frac{3}{2}{\xi_2}^2 - \frac{1}{2}, \xi_1\xi_2, \frac{3}{2}{\xi_1}^2 - \frac{1}{2}, \frac{5}{2}{\xi_2}^3 - \frac{3}{2}\xi_2, \\ & \xi_1 \cdot (\frac{3}{2}{\xi_2}^2 - \frac{1}{2}), (\frac{3}{2}{\xi_1}^2 - \frac{1}{2}) \cdot \xi_2, \frac{5}{2}{\xi_1}^3 - \frac{3}{2}\xi_1\Big\}. \end{aligned}$$

3 NUMERICAL RESULTS

To benchmark the proposed framework, we consider two test problems: the stochastic diffusion equation and the stochastic Helmholtz equation. In both problems, we show the convergence of SFEMs, associate with total degree and Euclidean degree of gPC basis in stochastic space, and a bi-linear finite element approximation in physical space.

To analyze the convergence of SFEMs, we define the relative errors of mean function and variance function:

$$\begin{aligned} \mathrm{err}_{\mathrm{mean}} &:= \frac{\|\mathbb{E}[u^{ap}] - \mathbb{E}[u_{\mathrm{ref}}]\|_2}{\|\mathbb{E}[u_{\mathrm{ref}}]\|_2}, \\ \mathrm{err}_{\mathrm{variance}} &:= \frac{\|\mathbb{V}[u^{ap}] - \mathbb{V}[u_{\mathrm{ref}}]\|_2}{\|\mathbb{V}[u_{\mathrm{ref}}]\|_2}, \end{aligned} \tag{9}$$

where u_{ref} is the reference solution equipped with maximal degree in the way of selecting basis functions.

3.1 Stochastic Diffusion Equation

Given a stochastic diffusion equation:

$$\begin{aligned} -\nabla \cdot \big(a(\boldsymbol{x}, \boldsymbol{\xi}) \nabla u(\boldsymbol{x}, \boldsymbol{\xi})\big) &= f(\boldsymbol{x}), \quad (\boldsymbol{x}, \boldsymbol{\xi}) \in D \times \Gamma, \\ u(\boldsymbol{x}, \boldsymbol{\xi}) &= g(\boldsymbol{x}), \quad (\boldsymbol{x}, \boldsymbol{\xi}) \in \partial D_D \times \Gamma, \\ a(\boldsymbol{x}, \boldsymbol{\xi}) \nabla u(\boldsymbol{x}, \boldsymbol{\xi}) \cdot \boldsymbol{n} &= h(\boldsymbol{x}), \quad (\boldsymbol{x}, \boldsymbol{\xi}) \in \partial D_N \times \Gamma, \end{aligned}$$

where the physical domain is $D = [-1, 1] \times [-1, 1]$, the boundary conditions denotes ∂D, Dirichlet boundary condition is $\partial D_D = [-1, 1] \times \{-1, 1\}$, and $\partial D_N = \partial D \backslash \partial D_D$. Given $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)$, and

$$\begin{aligned} a(\boldsymbol{x}, \boldsymbol{\xi}) &= 10.1 + \sum_{k=1}^{4} [2.25 + 0.25 \cos(k x_1)] \xi_k, \\ f(\boldsymbol{x}) &= \frac{2 - x_1^2 - x_2^2}{8}, \\ g(\boldsymbol{x}) &= x_1, \\ h(\boldsymbol{x}) &= e^{x_1 + x_2}. \end{aligned}$$

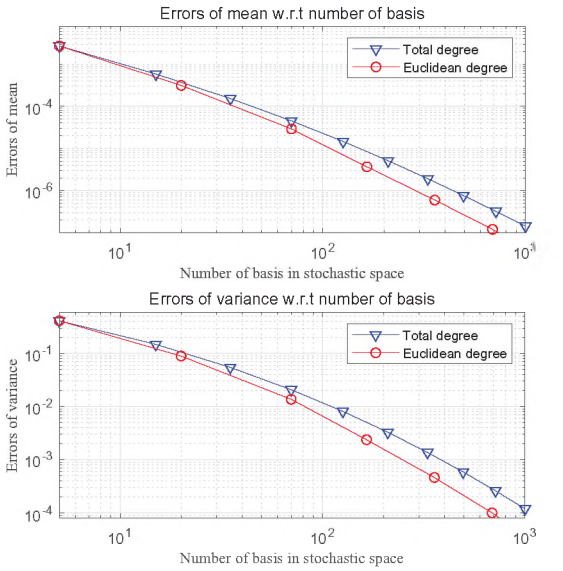

Figure 1: Comparison Of Errors For Basis Selected By The Euclidean Degree And Total Degree In Stochastic Space.

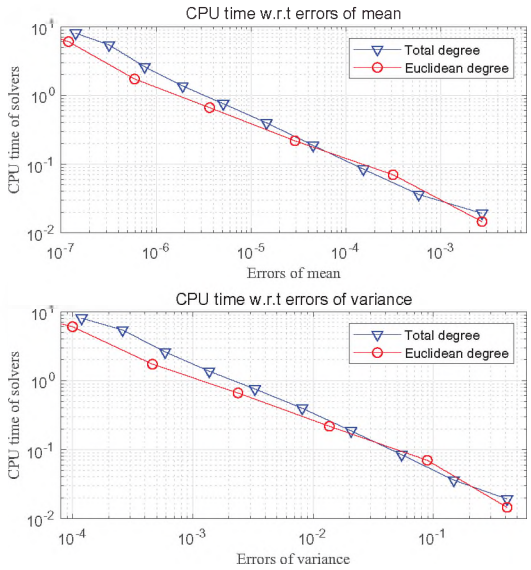

Figure 2: Comparison Of The CPU Time For Basis Selected By The Euclidean Degree And Total Degree In Stochastic Space.

Figure 1 shows the relative errors of mean and variance with respect to the number of basis, where the relative errors of mean function and variance function are given in (9), and the basis functions in stochastic space are selected by satisfying total degree and Euclidean degree. Besides, we apply the same uniform spatial grid in physical space, and the reference solution in (9) is obtained by gPC method for setting the maximal degree $d_M = 10$. The linear system (5) is solved by conjugate gradient. The upper one in figure 1 shows that the relative errors of mean decrease quickly as the number of basis increasing and the relative errors obtained by Euclidean degree is much quickly than by total degree and the relative errors of variance shows the same trend.

Figure 2 shows the CPU time with respect to errors of mean and variance of solution, where the CPU time includes the time for constructing the linear system (5) and solving that using conjugate gradient, the CPU time units are seconds. Besides, the mean and variance are obtained by SFEMs, where the basis functions in stochastic space are selected by satisfying total degree and Euclidean degree. The upper one in Figure 2 shows the CPU time increases as the errors mean going down, and the CPU time increases slower when the basis functions in stochastic space are selected by Euclidean degree than by total degree. Similarly, the lower one in Figure 2 shows the CPU time shows the same trend with respect to the relative errors of variance.

Table 2 shows the relative errors of mean and variance of solution with respect to the order of basis, where the basis functions are selected by satisfying the total degree and Euclidean degree. The reference solution is obtained by setting Maximal degree $d_M = 10$, and it is easily seen that the solver basis selected by Euclidean degree are much less than by total degree for the same accuracy.

3.2 Stochastic Helmholtz Equation

Given a stochastic Helmholtz equation:

$$-\nabla^2 u(\boldsymbol{x},\boldsymbol{\xi}) - \kappa^2(\boldsymbol{x},\boldsymbol{\xi})u(\boldsymbol{x},\boldsymbol{\xi}) = f(\boldsymbol{x}),\ (\boldsymbol{x},\boldsymbol{\xi}) \in D\times\Gamma$$
$$u(\boldsymbol{x},\boldsymbol{\xi}) = 0,\ (\boldsymbol{x},\boldsymbol{\xi}) \in \partial D_D \times \Gamma$$

where the physical domain is $D = [-1,1]\times[-1,1]$, and the pure Dirichlet boundary condition $\partial D = \partial D_D$. Given $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{\xi} = (\xi_1, \xi_2)$, and

$$\kappa(\boldsymbol{x},\boldsymbol{\xi}) = \frac{\sqrt{2}\pi}{2} + 0.41 + 0.18(\xi_1+\xi_2) + 0.02\big[\cos(x_1)\xi_1 + \sin(x_2)\xi_2\big],$$
$$f(\boldsymbol{x}) = \cos\left(\frac{\pi x_1}{2}\right)\cos\left(\frac{\pi x_2}{2}\right).$$

Figure 3 shows the relative errors of mean and variance with respect to the number of basis with the same setting as for solving stochastic diffusion equation except the maximal degree $d_M = 30$. Furthermore, the trend of the relative errors of mean function and variance function are similar to Figure 1 and we skip the analysis for it.

Figure 4 shows the CPU time with respect to errors of mean and variance of solution and the detail analysis can refer to that of Figure 1.

Table 3 shows the relative errors of mean and variance of solution with respect to the order of basis, where the basis functions are selected by satisfying the total degree and Euclidean degree. The reference solution is

Table 2: Error Estimator With Respect To Total Degree And Euclidean Degree

order of basis	err_{mean} of d_T	err_{mean} of d_E	$\text{err}_{\text{variance}}$ of d_T	$\text{err}_{\text{variance}}$ of d_E
1	0.0028	0.0028	0.4135	0.4135
2	6.0787×10^{-4}	3.2637×10^{-4}	0.1498	0.0903
3	1.5956×10^{-4}	3.0520×10^{-5}	0.0555	0.0140
4	4.7376×10^{-5}	3.8859×10^{-6}	0.0212	0.0025
5	1.5409×10^{-5}	6.3517×10^{-7}	0.0084	4.8319×10^{-4}
6	5.4051×10^{-6}	1.2881×10^{-7}	0.0034	1.0557×10^{-4}
7	2.0281×10^{-6}	3.3548×10^{-8}	0.0014	2.7387×10^{-5}
8	8.0997×10^{-7}	9.7839×10^{-9}	6.1533×10^{-4}	7.6549×10^{-6}
9	3.4305×10^{-7}	3.0671×10^{-9}	2.7398×10^{-4}	2.2730×10^{-6}
10	1.5349×10^{-7}	1.0437×10^{-9}	1.2576×10^{-4}	7.3632×10^{-7}

Table 3: Error Estimator With Respect To Total Degree And Euclidean Degree

order of basis	err_{mean} of d_T	err_{mean} of d_E	$\text{err}_{\text{variance}}$ of d_T	$\text{err}_{\text{variance}}$ of d_E
1	0.0713	0.0713	0.7356	0.7356
5	0.0017	6.3373×10^{-4}	0.0697	0.0314
10	2.3588×10^{-5}	1.8685×10^{-6}	0.0020	2.0412×10^{-4}
15	3.5431×10^{-7}	6.2159×10^{-9}	4.5019×10^{-5}	1.0530×10^{-6}
20	5.5264×10^{-9}	2.1489×10^{-11}	9.4271×10^{-7}	4.9309×10^{-9}
25	8.8226×10^{-11}	7.6249×10^{-14}	1.8888×10^{-8}	2.1913×10^{-11}
30	1.4316×10^{-12}	8.1489×10^{-16}	3.6860×10^{-10}	8.4303×10^{-14}

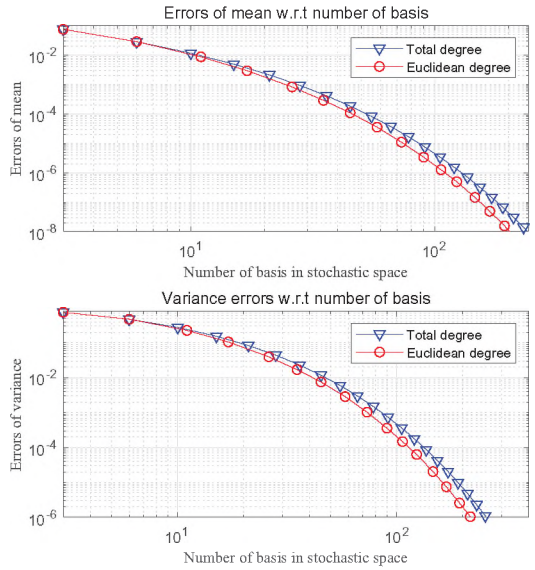

Figure 3: Comparison Of Errors For Basis Selected By The Euclidean Degree And Total Degree In Stochastic Space.

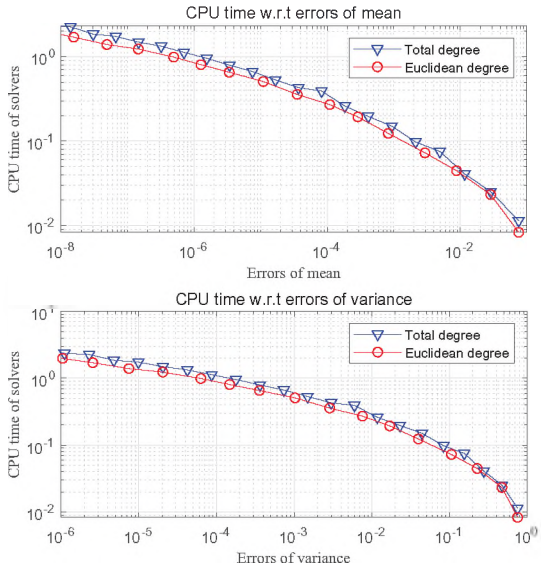

Figure 4: Comparison Of The CPU Time For Basis Selected By The Euclidean Degree And Total Degree In Stochastic Space.

obtained by setting Maximal degree $d_M = 30$, and it is easily seen that the solver basis selected by Euclidean degree are much less than by total degree for the same accuracy.

4 CONCLUSIONS

We have proposed a new implementation of gPC methods to improve the convergence rate when solving PDEs with random inputs. Our approach selects the basis functions by satisfying the Euclidean degree. The linear system is obtained through stochastic finite element methods and with the special properties, like the sparse and symmetric matrix in linear system, our algorithm employes the standard conjugate gradient to solve it. In the case studies, the performances demonstrate that our method is able to accelerate convergence in both two test problems by the Euclidean degree.

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