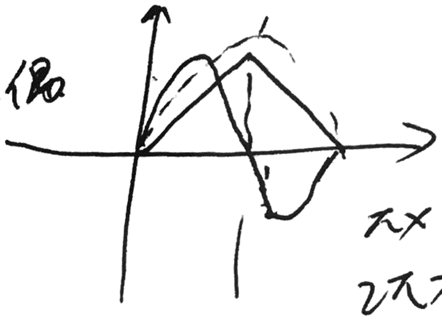
(2.1) ①

$$a_m = 2\int_0^1 u^0(x)\sin(m\pi x)dx \quad m=偶$$



$$= 4\int_0^{\frac{1}{2}} 2x \sin m\pi x\, dx$$

$$= 8\int_0^{\frac{1}{2}} x \sin m\pi x\, dx$$

$$= -\frac{8}{m\pi}\int_0^{\frac{1}{2}} x\, d\cos(m\pi x)$$

$$= -\frac{8}{m\pi}\Big[\underbrace{x\cos m\pi x\Big|_0^{\frac{1}{2}}}_{0} - \int_0^{\frac{1}{2}}\cos m\pi x\, dx\Big]$$

$$= \frac{8}{m\pi}\int_0^{\frac{1}{2}}\cos m\pi x\, dx$$

$$= \frac{8}{m\pi}\cdot\frac{1}{m\pi}\sin m\pi x\Big|_0^{\frac{1}{2}}$$

$$= \frac{8}{m^2\pi^2}\sin\frac{m\pi}{2} \quad m=偶$$

$\underline{m=奇 \Rightarrow} a_m = 0.$

$$\Rightarrow a_m = \frac{8}{m^2\pi^2}\sin\frac{m\pi}{2}\,..$$

②

$$\int_{2p}^{2p+2}\frac{1}{x^2}dx = -\frac{1}{x}\Big|_{2p}^{2p+2} = \frac{1}{2p} - \frac{1}{2p+2} = \frac{2}{(2p+2)2p}$$

$2p\cdot(2p+2) \le (2p+1)^2$. $2p=2p+2$ 时取等 取不到

故 $2p\cdot(2p+2) < (2p+1)^2$. $\int_{2p}^{2p+2}\frac{1}{x^2}dx > \frac{2}{(2p+1)^2}$.

$$\sum_{p=p_0}^{\infty}\frac{1}{(2p+1)^2} < \frac{1}{2}\Big[\int_{2p_0}^{2p_0+2} + \int_{2p_0+2}^{2p_0+4} + \cdots\Big] = \frac{1}{2}\cdot\frac{1}{2p_0} = \frac{1}{4p_0}.$$

(II) $\left| E \right| = \left| \sum_{m=2p_0+1}^{\infty} a_m \sin m\pi x \right|$ $\quad \dfrac{2p_0+1}{p_0}$

$\leq \dfrac{8}{\pi^2}\left[\dfrac{1}{(2p_0+3)^2} + \ldots + \dfrac{1}{(2p_0+\ldots)^2} \ldots \right]$

$\cancel{4}.203$

$< \dfrac{8}{\pi^2} \cdot \dfrac{1}{4(p_0+1)} = \dfrac{2}{\pi^2 \cdot (p_0+1)} = 9.98 \times 10^{-4}$

2.2 ① $\exp(-k^2 \Delta t) = 1 - k^2 \Delta t + \frac{1}{2} k^4 (\Delta t)^2 + \ldots \ldots$

$\lambda(k) = 1 - 4\mu \sin^2 \frac{1}{2} k \Delta x = 1 - 2\mu(1 - \cos k\Delta x)$ (2.51)

$= 1 - 2\mu \left[\frac{1}{2}(k\Delta x)^2 - \frac{1}{24}(k\Delta x)^4 + \ldots \right]$

$= 1 - k^2 \Delta t + \frac{1}{12} k^4 \Delta t (\Delta x)^2$

$= 1 - k^2 \Delta t + \frac{1}{12} k^4 \cdot \Delta t^2 \cdot \frac{1}{\mu}$

$$\frac{|\lambda(k) - \exp(-k^2\Delta t)|}{k^4 (\Delta t)^2} = \underbrace{\left| \frac{1}{2} - \frac{1}{12}\frac{1}{\mu} \right|}_{C(\mu)}$$

$\mu = \frac{1}{4}$, $\cancel{C(\tfrac{1}{4})} = \left| \frac{1}{2} - \frac{1}{12} \cdot 4 \right| = \left| \frac{1}{2} - \frac{1}{3} \right|$ $< \frac{1}{2}$

(II) $\sum_{m=2p_0+1}^{\infty} |a_m| < \frac{1}{4p_0} \cdot \cancel{\frac{1}{4p_0}} \cdot \frac{8}{\pi^2}$

$= \frac{1}{4} \cdot \frac{8}{\pi^2 \cdot p_0}$ $\quad \boxed{\frac{8}{\pi^2 p_0} = 0.0099}$

$< \frac{1}{4} \varepsilon$.

$\sum_{m=1}^{2p_0-1} |a_m| m^4 = \frac{8}{\pi^2} \left[1^2 + 3^2 + 5^2 + \ldots + (2p_0-1)^2 \right]$

$$= \frac{8}{\pi^2}\left[1^2+2^2+\dots+(2p_0-1)^2 - 2^2-4^2-\dots-(2p_0-2)^2\right]$$

$$= \frac{8}{\pi^2}\cdot \frac{(2p_0-1)\,\cancel{4}p_0\,(4p_0-1)}{\cancel{6}\;3} - \frac{\cancel{4}^{2}(p_0-1)\,p_0\,(2p_0-1)}{\cancel{6}\;3} \qquad \sum_1^{n} n^2 = \frac{n(n+1)(2n+1)}{2}$$

$$= \frac{8}{3\pi^2}\cdot p_0\,(2p_0-1)\,[\underbrace{(4p_0-1) - 2(p_0-1)}_{4p_0-1-2p_0+2}]$$

$$= \frac{8}{3\pi^2}\cdot p_0\cdot(2p_0-1)(2p_0+1).$$

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by p.62)

$$|e_j^n| \le \frac{1}{2}\varepsilon + \underset{1}{\underline{\underline{t_T}}}\cdot\frac{1}{2}\cdot\pi^4\cdot \frac{8p_0\cdot(2p_0+1)(2p_0-1)}{3\pi^2}\cdot 1.7\times10^{-10}$$

$$= \frac{1}{2}\varepsilon + \frac{1}{2}\cdot\frac{1}{3}\pi^2 \times 1.7\times10^{-10}$$

$$= \frac{1}{2}\varepsilon + \frac{1}{2}\cdot 0.0099 < \varepsilon.$$

(2b) ① $b^2 \le 4c \Rightarrow |x_1|^2 = |x_1 x_2| = |c| < 1$

$$\Rightarrow b^2 \le 4c \le |1+c|^2.$$

$b^2 > 4c$ $\quad |x_1 x_2| = |c| < 1$

$$-1 \le \frac{-b \pm \sqrt{b^2-4c}}{2} \le 1 \Rightarrow \begin{cases} (\sqrt{b^2-4c})^2 \le (2+b)^2 \\ (\sqrt{b^2-4c})^2 \le (2-b)^2 \end{cases}$$

$$\Rightarrow \begin{cases} b \ge \underline{-1-c} \\ b \le 1+c \end{cases} \Rightarrow |b| \le 1+c.$$

$1+c > 0$

② $\delta_x^2 u_j^n = U_{j+1}^n - 2U_j^n + U_{j-1}^n$

代入 $U_j^{n+1} = \lambda U_j^n \qquad U_j^n = (\lambda)^n e^{ik(j\Delta x)}$

$$\Rightarrow \lambda^2 U_j^{n-1} - U_j^{n-1} = \frac{1}{3}\mu(\lambda^2+\lambda+1)(\delta_x^2 U_j^{n-1})$$

$$\Rightarrow (\lambda^2-1)U_j^{n-1} = \frac{1}{3}\mu(\lambda^2+\lambda+1)U_j^{n-1}(e^{ik\Delta x} - 2 + e^{-ik\Delta x})$$

$$\Rightarrow (\lambda^2-1) = \frac{1}{3}\mu(\lambda^2+\lambda+1)\cdot[2\,\cancel{\cos}\,\underbrace{(1-\cos k\Delta x)}_{a}]$$

$$\Rightarrow (1+\frac{2}{3}\mu a)\lambda^2 + \frac{2}{3}\mu a\lambda + (\frac{2}{3}\mu a - 1) = 0$$

$$\lambda^2 + \frac{\frac{2}{3}\mu a}{\underline{1+\frac{2}{3}\mu a}}\lambda + \underbrace{\frac{\frac{2}{3}\mu a - 1}{1+\frac{2}{3}\mu a}}_{<1} = 0$$

$|b| \le 1+c$ ok.

(III) $(\lambda^2-1)v_j^{n-1} = \frac{1}{6}\mu(\lambda^2+4\lambda+1)[-2(1-\cos k\Delta x)]v_j^{n-1}$

$$\Rightarrow (1+\tfrac{1}{3}\mu a)\lambda^2 + \tfrac{4}{3}\mu a\lambda + (\tfrac{1}{3}\mu a-1) = 0$$

$$\Rightarrow \lambda^2 + \frac{\frac{4}{3}\mu a}{1+\frac{1}{3}\mu a}\lambda + \frac{\frac{1}{3}\mu a-1}{1+\frac{1}{3}\mu a} = 0$$

$$1+c = \frac{\frac{1}{3}\mu a-1+1+\frac{1}{3}\mu a}{1+\frac{1}{3}\mu a} = \frac{\frac{2}{3}\mu a}{1+\frac{1}{3}\mu a} < b$$

故 ~~|根|~~ $|\lambda|>1$ unstable.